COMMUNICATIONS TO THE EDITOR

Countercurrent Heat or Mass Transfer Between a Turbulent and a Laminar Stream: I. Flat Velocity Profiles and Short Contact Times

E. N. LIGHTFOOT
University of Wisconsin, Madison, Wisconsin

Presented below is the description of a simple, countercurrent apparatus in which heat or mass is transferred between a turbulent and a laminar stream. Flat velocity profiles in the laminar stream and short contact times are assumed. This analysis differs from those previously published (*1, 5*) primarily in that an energy or mass balance is used as a boundary condition for the laminar stream, rather than a predetermined wall temperature or wall heat flux distribution. The boundary condition used here is that most commonly applicable to practical heat or mass transfer equipment, whereas the previously available analyses are applicable only under special circumstances, for example when the flow rate and/or conductivity of the turbulent stream are very large. The results presented below show that the effects of relative stream capacities and conductivities cannot always be reliably estimated from presently available correlations for transfer coefficients in the individual streams. It therefore seems desirable that this analysis be extended to consideration of long contact times and other velocity profiles.

SYSTEM CONSIDERED

Consider here steady state transfer of heat or mass between two countercurrent streams, one laminar and one turbulent. The laminar stream is assumed to have a flat velocity profile and to enter the system with a uniform temperature or composition. Transfer to or from the turbulent stream is assumed describable in terms of a constant local heat or mass transfer coefficient h_{loc} or $k_{c_{loc}}$. In addition contact time between the two streams is considered to be sufficiently short so that temperature or concentration changes in the laminar fluid are limited to a small region adjacent to the turbulent fluid. Such a system is described by the following differential equations and boundary conditions:

Laminar stream

$$\frac{\partial \Theta_l}{\partial \zeta} = 2\frac{\partial^2 \Theta_l}{\partial \eta^2} \tag{1}$$

At

$$\zeta = 0,\ \Theta_l = 0 \tag{2}$$

At

$$\eta = 0,\ \frac{\partial \Theta_l}{\partial \eta} = \beta\,(\Theta_l - \Theta_t) \tag{3}$$

At

$$\eta = \infty,\ \Theta_l = \text{finite} \tag{4}$$

Turbulent stream

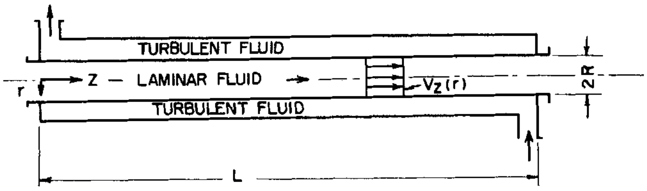
$$-\frac{d\,\Theta_t}{d\,\zeta} = \alpha \left.\frac{\partial\,\Theta_l}{\partial \eta}\right|_{\eta=0} \tag{5}$$

At

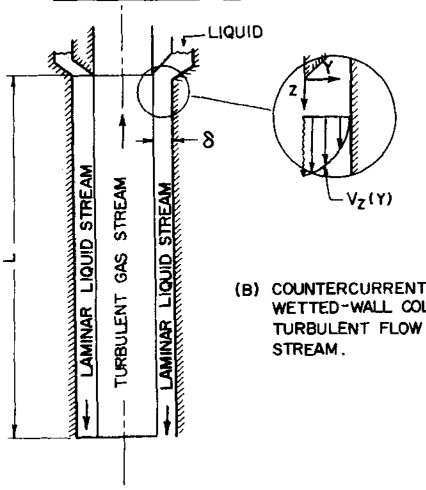
$$\zeta = 0,\ \Theta_t = 1 \tag{6}$$

Here $\Theta_l(\eta, \zeta)$ and $\Theta_t(\zeta)$ are the reduced temperatures or compositions of the laminar and turbulent streams, respectively. These quantities are explicitly defined for two physical systems in Figure 1 and Table 1. The mass transfer system considered is quite realistic; the heat transfer system represents a limiting case, approached for highly pseudoplastic fluids or for normal liquids being rapidly heated.

To describe the rate of transfer between the two streams it is only necessary to know the bulk temperature or composition of the turbulent stream as a function of position, that is $\Theta_t(\zeta)$. This information may be obtained directly through use of the Laplace transformation. One thus obtains



(A) COUNTERCURRENT HEAT TRANSFER BETWEEN A LAMINAR STREAM IN A ROUND TUBE AND A TURBULENT STREAM VELOCITY PROFILE OF THE LAMINAR STREAM IS FLAT.



(B) COUNTERCURRENT FLOW IN A SHORT WETTED-WALL COLUMN WITH TURBULENT FLOW IN THE GAS STREAM.

Fig. 1. Representative physical situations described by Equations (1) through (6).

TABLE 1. DEFINITIONS OF THE REDUCED PARAMETERS FOR THE PHYSICAL SITUATIONS OF FIGURE 1

Reduced parameter	Definition: Heat transfer	Definition: Mass transfer
Θ_l	$(T_l - T_{lo})/(T_{to} - T_{lo})$	$(c_l - c_{lo})/(c_t/m - c_{lo})$
Θ_t	$(T_t - T_{lo})/(T_{to} - T_{lo})$	$(C_t/m - c_{lo})/(c_t/m - c_{to})$
ζ	$\pi k_l z/2w_l C_{pl}$	$2z\mathcal{D}_{AL}/v_{max}\delta^2$
λ	$\pi k_l L/2w_l C_{pl}$	$2L\mathcal{D}_{AL}/v_{max}\delta^2$
η	$1 - \dfrac{r}{R}$	Y/δ
β	$h_{loc}R/k_l$	$k_{c,\,loc}\delta/\mathcal{D}_{AL}$
α	$-4w_l C_{pl}/w_t C_{pt}$	$-v_{max}\delta/mv_t R$

$$\Theta_t = \left(\frac{R+1}{2R}\right) e^{(R-1)^2 Z} \left[1 + \operatorname{erf}\,(R-1)\sqrt{Z}\right] + \left(\frac{R-1}{2R}\right) e^{(R+1)^2 Z} \left[1 - \operatorname{erf}\,(R+1)\sqrt{Z}\right] \quad (7)$$

where

$$R = 1 + \alpha/\beta \quad (8)$$

$$Z = \beta^2 \zeta/2 \quad (9)$$

Equation (7) is valid for all α, β, and ζ, within the limits of Equations (1) through (6), but it is frequently inconvenient to use. It is therefore useful to consider the following special cases of this equation:

$$\lim_{\beta\to\infty} (\Theta_t) = 1 + \alpha\sqrt{2\,\zeta/\pi} \quad (10)$$

Small α

$$\lim_{\beta\to\infty} (\Theta_t) = [1 + \operatorname{erf}\,(\alpha\sqrt{\zeta/2})]e^{\alpha^2\zeta/2} \quad (11)$$

All α

$$\lim (\Theta_t) = 1 + \frac{2\alpha}{\pi\beta} \cdot g(\beta\sqrt{\pi\,\zeta/2}) \quad (12)$$

Small α

All β

where

$$g(\gamma) = e^{\gamma^2}(1 - \operatorname{erf}\gamma) + \frac{2\gamma}{\pi} - 1 \quad (13)$$

Equations (10) and (12) can also be obtained directly from published solutions to other heat or mass transfer problems (*1, 2*). The laminar temperature profile will not be calculated in this paper as it is of no utility for our present purposes.

BEHAVIOR OF SYSTEM

For convenience in working practical problems Equations (7) through (13) were used to calculate dimensionless overall heat or mass transfer coefficients:

$$N_{Nu} = \frac{2}{\alpha\lambda}[\Theta_t(\lambda) - 1] \quad (14)$$

These coefficients are based on the driving forces at the laminar inlet. Thus for heat transfer (*3*)

$$N_{Nu} = U_1 D/k_l \quad (15)$$

$$\lim_{\beta\to\infty} (N_{Nu}) = h_1 D/k_l \quad (16)$$

For mass transfer N_{Nu} is defined analogously in terms of mass transfer coefficients and diffusivity of the laminar fluid.

The effect of relative stream capacities, that is of α, on the overall coefficient is shown in Figure 2. Here N_{Nu} is plotted as a function of reduced exchanger length λ (ζ at laminar exit) for $\beta = \infty$ and several α. Also shown for comparison is the solution for $\alpha = 0$ and long contact times (*1*) (Curve A).* The difference between curves A and B represents the approximation introduced by use of Equation (4). Curve D, for $\alpha = 4$, represents system behavior when $(\Theta_l - \Theta_t)$ is constant.

The interaction of the two streams, that is the effect of β, is shown in Table 2 in terms of

$$r = \frac{\left[\dfrac{1}{N_{Nu_t}} + \dfrac{1}{N_{Nu}(\alpha, \infty, \lambda)}\right]}{[1/N_{Nu}(\alpha, \beta, \lambda)]} \quad (17)$$

Here N_{Nu_t} is the value of N_{Nu} for the existing turbulent transfer coefficient but for infinite conductivity in the laminar stream. Then $1/N_{Nu}(\alpha, \infty, \lambda)$ represents the resistance of the laminar stream, for the case of zero resistance in the turbulent stream, and $(1/N_{Nu_t})$ represents the resistance of the turbulent stream. Both of these quantities can be estimated from published correlations (*5*)†. The quantity $1/N_{Nu}(\alpha, \beta, \lambda)$ represents the combined resistance of the two streams; it could be obtained merely by adding the separately calculated stream resistances if r were constant at unity.

From Figure 2 and Table 2 one finds that both α and β have a pronounced effect on N_{Nu} and also on the variation of N_{Nu} with λ:

1. As α increases N_{Nu} increases, and the effect of α is greater for larger λ, that is for longer exchangers. The errors introduced by use of Equation (4) also appear to extend to lower λ for larger α, but this point cannot be proven until a more general solution is available.
2. The total transfer resistance of the two streams is less than the sum of the separately calculated resistances of each.

These conclusions make it appear worthwhile to develop expressions corresponding to Equation (7) for long exchangers and other velocity profiles.

* For the geometry of Figure 1*a*, Curve A of Figure 2 is obtained by replacement of Equation (4) by the relation

$$\text{At} \quad \eta = 1 \quad \partial\Theta_l/\partial\eta = 0 \quad (4a)$$

Also for this geometry $\Phi = N_{Re}N_{Pr}D/L$ (heat transfer), or $N_{Re}N_{Sc}D/L$ (mass transfer)

† For the calculation of $(N_{Nu})_t$ see reference 5. For calculation of $N_{Nu}(\alpha, \infty, \lambda)$ see Equation (11), this paper. For calculation of $N_{Nu}(o, \infty, \lambda)$ see Equations (10) and (14), this paper, or reference *1*.

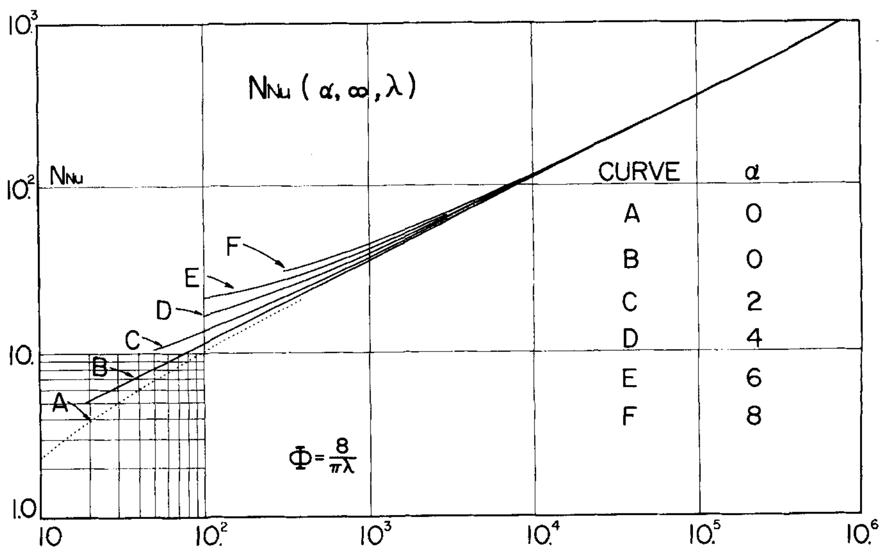


Fig. 2. The effect of relative stream capacities on overall transfer coefficients for $\beta = \infty$. Curves B through F represent solutions of Equations (1) through (6) for $\beta = \infty$ and the indicated α. Curve A represents the Graetz solution for $\alpha = 0$, $\beta = \infty$, and long contact times.

Table 2. The Effect of β on Transfer Coefficients

λ	$N_{Nu}(\alpha, \infty)$	$N_{Nu}(\alpha, \beta)$	$N_{Nu}(\alpha, \infty)/2\beta$	N_{Nu}/N_{Nuapp}	α	β
10^6	1,128.4	756.0	0.5642	1.048	0	1,000.0
10^4	112.84	108.0	0.05642	1.011	0	1,000.0
10^2	11.284	11.23	0.005642	1.001	0	1,000.0
10^6	1,132.0	757.5	0.5660	1.048	4	1,000.0
10^4	117.0	111.5	0.05848	1.009	4	1,000.0
10^2	16.90	16.77		1.001	4	1,000.0
10^6	1,132.0	556.6	1.132	1.048	4	500.0
10^4	117.0	106.13	0.1170	1.013	4	500.0
10^2	16.90	16.64	0.01690	1.001	4	500.0

$$N_{Nuapp} = \left[\frac{1}{N_{Nu}(\alpha, \infty)} + \frac{1}{2\beta}\right]^{-1}$$

However available correlations, and the assumption of additivity of separately calculated resistances, will be satisfactory for many purposes and will probably lead to conservative design in most cases.

NOTATION

D = equivalent diameter of laminar stream

h_l = average heat transfer coefficient for the laminar stream, based on inlet temperature difference

k_l = thermal conductivity of fluid in laminar stream

U_l = corresponding overall heat transfer coefficient

Greek Letters

α = measure of relative capacities

β = transfer resistances of two streams

ζ = reduced distance in the direction of flow of the laminar stream

η = reduced distance into laminar stream

λ = reduced length of apparatus, that is the value of ζ at the laminar-stream outlet

LITERATURE CITED

1. Drew, T. B., *Trans. Am. Inst. Chem. Engrs.*, **26**, 26-80 (1931).
2. Crank, J., "The Mathematics of Diffusion," Oxford Univ. Press, London, England (1956).
3. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," Wiley, New York (1960).
4. Siegel, R., E. M. Sparrow, and F. M. Hallman, *Appl. Sci. Research,* **A7**, 386-92 (1958).
5. Jakob, Max, "Heat Transfer," Vol. 1, pp. 547-554, Wiley, New York (1949).

A Test of the Assumption of Interfacial Equilibrium in Measurements of the Gas Film Mass Transfer Coefficient

L. J. DELANEY and L. C. EAGLETON

University of Pennsylvania, Philadelphia, Pennsylvania

Rates of interphase mass transfer have attracted the attention of most chemical engineers at one time or another. Concurrent with this interest has been the concern with the assumption of interfacial equilibrium at the phase boundary where the transfer of mass is occurring. It has usually been assumed that the concentrations existing at a gas-liquid interface where mass transfer occurs are those that would be present if no mass were transferring and the system were maintained at the same conditions of temperature and pressure. However most of the concern has been qualitative for two main reasons, namely the deviations were expected to be small, if not undetectable, and the deviation itself depended on a quantity which was somewhat in doubt, the condensation coefficient α. The recent activity in this field among chemical engineers (*3, 9, 10, 11, 13*) confirms the interest in the problem. The following is an attempt to make a quantitative evaluation of the deviations from interfacial equilibrium in the case of vaporization of pure liquids and solids into a foreign gas.

THEORY

The basis of the discussion of rates of vaporization of pure liquids is the Hertz-Knudsen equation

$$\frac{dn}{dt} = \frac{p}{(2\pi\, m\, kT)^{1/2}} \qquad (1)$$

where $(dn)/(dt)$ is the number of molecules per unit time passing through an arbitrary plane of 1 sq. cm. area randomly placed in the gas phase in which the molecules have a Maxwellian distribution. If the reference plane is placed at the gas-liquid interface, Equation (1) with $p = p_i$ gives the number of molecules striking the liquid surface from the gas phase. At equilibrium this also gives the number of molecules leaving the surface by reflection and by evaporation. If one assumes that a certain fraction α of the molecules striking the liquid surface actually condenses, then at equilibrium $\alpha\, p_s/(2\pi\, m\, kT)^{1/2}$ molecules enter and leave the liquid per square centimeter of surface per second. This expression is a function only of the nature of the liquid surface and will represent the rate that molecules leave the liquid under nonequilibrium conditions as well as at equilibrium. The net flux N_A will then be given by the difference in the opposing rates of mass transfer:

$$N_A = \frac{\alpha P_s}{(2\pi MRT_s)^{1/2}} - \frac{\alpha P_i}{(2\pi MRT_i)^{1/2}} \approx \frac{\alpha(P_s - P_i)}{(2\pi MRT)^{1/2}} \qquad (2)$$